to within less than 3 per cent error over the entire range covered. Since the absorption spectrum contains transparent regions in which radiation does not influence the temperature profile, the improved accuracy is to be expected. This finding is significant since it demonstrates that results of engineering accuracy for realistic materials can be predicted.

CONCLUSIONS

The two-term Taylor series expansion of the emissive power has been shown to be an accurate and useful method for the prediction of the net heat transfer for combined radiation and conduction. The approximation leads to a meaningful definition of the radiative conductivity which is extremely useful when radiation is coupled with other modes of heat transfer since the radiation can then be treated as a diffusion process with heat sources and sinks. This "effective" conductivity has a wider range of applicability than the classical radiative conductivity since it accounts for spectral as well as wall effects and is exact in the transparent and optically thick limits. It has been shown to accurately predict the total heat flux under a variety of boundary conditions and for nongray as well as gray media.

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THIN LIQUID FILMS UNDER SIMULTANEOUS SHEAR AND GRAVITY FORCES

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NOMENCLATURE

a, acceleration due to body forces;
Re_L, film Reynolds number $4\Gamma/\mu_L$;
u, local liquid film velocity;
u^+, u/u_*;
u_*, shear velocity $(\tau_w/\rho_L)^{\frac{1}{2}}$;
u_{*i}, interfacial shear velocity $(\tau_i/\rho_L)^{\frac{1}{2}}$;
y, distance from wall;
y^+, yu_*/ν_L;
β, shear parameter $(u_{*i}^3/a\nu_L)^{\frac{1}{3}}$;
Γ, mass flow rate per unit wetted perimeter;
δ, mean film thickness;
δ^+, non-dimensional film thickness $\delta u_*/\nu_L$;
η, non-dimensional film thickness $\delta(a/\nu_L^2)^{\frac{1}{3}}$;
μ_L, dynamic viscosity of liquid;
ν_L, kinematic viscosity of liquid;
ρ_L, ρ_v, density of liquid and vapor respectively;
τ_w, wall shear;
τ_i, interfacial shear.

INTRODUCTION

A common feature of many two phase processes involves the transport of heat or of mass across a flowing thin liquid film. One may cite climbing film evaporators, condensers, boiling-water nuclear reactors and some desalination processes as examples. The central problem is to predict the duty of such equipment. The transport resistance may be set up from the appropriate conservation equations by integrating the profile of interest across the film. As examples of this procedure Rohsenow *et al.* [1] and Altman *et al.* [2] have solved certain condensation problems using this technique while Dukler [3] has solved some falling film problems in this way.

The central problem is the computation of the film thickness from available variables. This film is unstable and waves are present on its surface (e.g. [5]) although it is normal to consider only the mean film thickness. This practice is followed here.

There are two analyses of film flow which have had considerable success [3, 4, 6]. Dukler's [3, 4] is set up primarily for falling film flow with vapor shear at the liquid–gas interface whereas Kutateladze [6] has set up various relationships. Dukler's results may be expressed in terms of the following quantities

$$\eta = fn\{Re_L, \beta\} \tag{1}$$

where

$$\eta = \delta(a/v_L^2)^{\frac{1}{3}}$$

and

$$Re_L = 4\Gamma/\mu_L$$

and

$$\beta = (u_{*i}^3/av_L)^{\frac{2}{3}}.$$

The complicated function of η (of two variables) requires that any results be interpreted graphically (or by computer) —see Fig. 1. Dukler's results should be extremely good close to the wall as well as good for thicker films because the velocity profile due to Deissler [7] was utilized. Note, as expected, that the well-known laminar analysis with zero interfacial shear due to Nusselt is progressively inaccurate as the Reynolds number increases.

On the other hand Kutateladze [6] had achieved a very simple result by integrating the logarithmic profile across the film assuming a two region turbulent flow analysis.

$$\delta^+(12{\cdot}0 + 10{\cdot}0 \ln \delta^+) - 156 = Re_L \tag{2}$$

where

$$\delta^+ = \delta u_*/v_L.$$

As would be expected this result is increasingly in error for small values of δ^+ but is good for "thick" films, $\delta^+ > \sim 10$. In addition, equation (2), although a very simple form, requires a trial and error calculation for the determination of δ^+. This is a minor nuisance for both hand calculations and for inclusion into explicit forms of the related transport equations of heat and mass.

ANALYSIS

For steady flowing liquid film with a parallel flow of high velocity vapor acting on it one may approximately write [1]

$$\tau_w = (\rho_L - \rho_v)\, a\delta \sin\theta \pm \tau_i \tag{3}$$

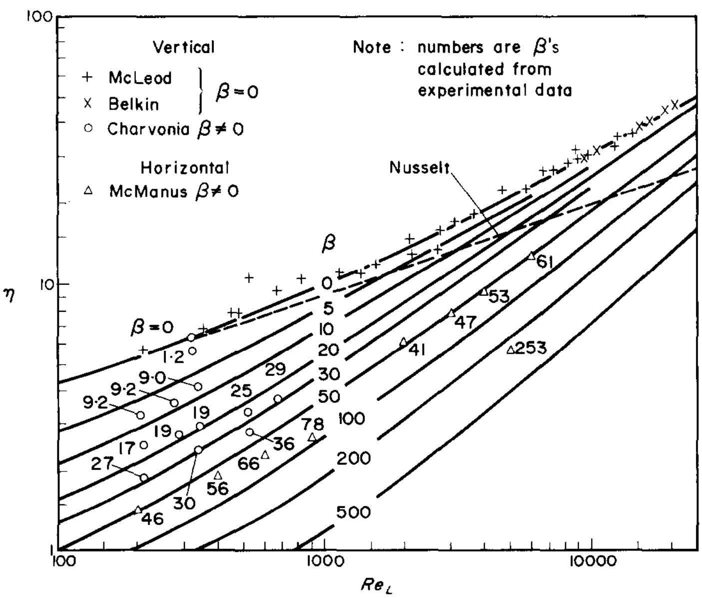

FIG. 1. Original film thickness analysis of Dukler.

or

$$u_*^2 = \left(\frac{\rho_L - \rho_v}{\rho_L}\right) a\delta \sin\theta \pm u_{*i}^2. \qquad (4)$$

The positive sign is for co-current flow and the negative sign is for countercurrent flow. In the analysis which follows it will become evident that only films in which the velocity distribution is a monotonic function of the distance from the wall can be accommodated. Thus we can only deal with co-current flow. This eliminates the negative sign in equation (4) for subsequent analysis and restricts θ in the range $0 > \theta > \pi/2$. Of course, at high shear rates compared to body force terms $u_* = u_{*i}$ and the orientation is irrelevant.

Note that there is a simple relationship between the non-dimensional film thickness used by Kutateladze and that used by Dukler. From equation (4) and the definition of δ^+

$$\delta^+ = \eta \left\{\left(\frac{\rho_L - \rho_v}{\rho_L}\right)\eta \sin\theta + \beta\right\}^{\frac{1}{2}} \qquad (5a)$$

or

$$\delta^+ \approx \eta(\eta \sin\theta + \beta)^{\frac{1}{2}}. \qquad (5b)$$

A simple alternative is now proposed to equations (1) and (2) which combines the advantages of algebraic simplicity and of reasonable accuracy for a full range of δ^+ values.

The mass flow rate per unit wetted perimeter of the thin liquid film is approximately given by

$$\Gamma = \rho_L \int_0^\delta u \, dy. \qquad (6)$$

From the definition of the film flow Reynolds number, $Re_L = 4\Gamma/\mu_L$ and equation (6) one can simply derive

$$Re_L = 4 \int_0^{\delta+} u^+ \, dy^+. \qquad (7)$$

The velocity distribution near to the wall is assumed to be

$$u^+ = y^+ \qquad (8)$$

(which is normally assumed to be accurate in pipe flow for $y^+ < 5$).

Hence from equations (7) and (8)

$$\delta^+ = (Re_L/2)^{\frac{1}{2}}. \qquad (9)$$

This expression is plotted in Fig. 2 against considerable experimental data [8–13]. For $\delta^+ < \sim 25$ ($Re_L < \sim 1000$) good agreement to data is noted.

For thicker films, rather than follow the method used by Kutateladze, one may proceed using Prandtl's $\frac{1}{7}$th power law velocity profile. This computation was originally derived in a specific non-general form by circuitous arguments in [2]. A straight forward derivation is now given.

The $\frac{1}{7}$th power law may be easily expressed as

$$u^+ = 8{\cdot}74 y^{+\frac{1}{7}}. \qquad (10)$$

Substitution of equation (10) into (7) and integrating leads to

$$\delta^+ = 0{\cdot}0504 \, Re_L^{\frac{7}{8}}. \qquad (11)$$

Equation (11) is also plotted in Fig. 2 from which it can be seen that excellent agreement to data is found for

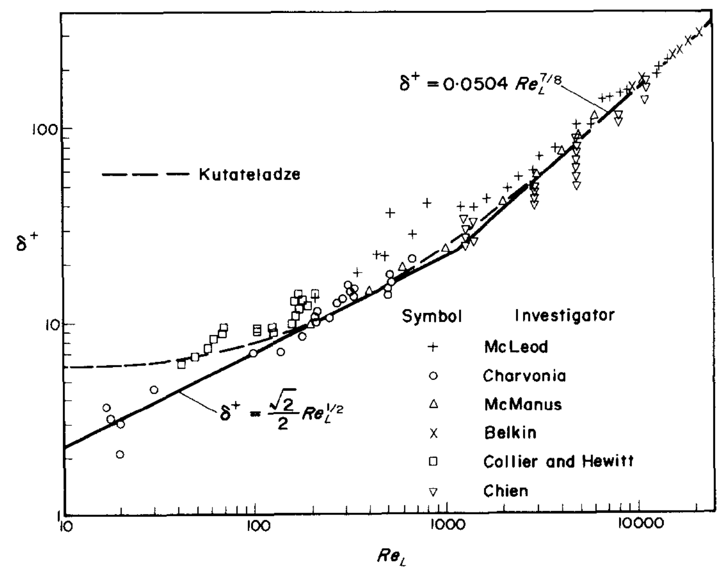

FIG. 2. Comparison of film thickness data to simple theories.

$\delta^+ > \sim 25$, $Re_L > \sim 1000$ except for the data of Chien [9] in which a significant spread occurs.

DISCUSSION

The non-dimensional film thickness, δ^+, plotted in Fig. 2 includes data of several authors including all those to which Dukler and Wicks [4] compared Dukler's theory given in Fig. 1. This includes a β range of 0–253 as determined from the original data. Those data were transformed from the map in Fig. 1 to Fig. 2 by use of the transformation given in equation (5b) with $\theta = \pi/2$. u_* was defined from equation (4) to include both interfacial shear terms and gravity terms. Thus the complicated function given in symbolic form as equation (1) and graphically in Fig. 1 may be reduced to a single curve of the form

$$\delta^+ = \delta^+(Re_L). \qquad (12)$$

The specific form of equation (12) may be simply approximated by a two region analysis

$$\delta^+ = (Re_L/2)^{\frac{1}{2}} \quad \text{for} \quad \delta^+ < 25 \qquad (13a)$$

$$\delta^+ = 0{\cdot}0504\, Re_L^{\frac{7}{8}} \quad \text{for} \quad \delta^+ > 25. \qquad (13b)$$

This first part of the result is surprising insofar as the basic velocity profile utilized should be only accurate to $y^+ < 5$. The insensitivity is due in part to the averaging process inherent in the integration process. Furthermore, the linear profile $u^+ = y^+$ ignores the zero interfacial shear boundary condition inherent in purely body force dominated flow situations. In laminar flow of course, this leads to a parabolic velocity profile and

$$\delta^+ = (\tfrac{3}{4} Re_L)^{\frac{1}{2}} \qquad (14)$$

as opposed to equation (13a). This result is 22 per cent higher than the line plotted for low Re_L in Fig. 2. However, Kapitza [14] has shown that rippling (which certainly occurs) leads to a thinner film than predicted by equation (14) so that a result between the predictions of equations (13a) and (14) may be most realistic.

Finally on Fig. 2, equation (2) due to Kutateladze [6] has been plotted. As expected inaccuracies occur for $\delta^+ < \sim 11{\cdot}6$. The same conclusion was reached by Kunz and Yarazunis [15] in their studies of transport through thin hydrodynamic films. Fundamentally this is due to the fact that the logarithic velocity profile utilized diverges from the true velocity as one approaches a solid boundary. Kunz and Yarazunis's result [15] is coincident with both of the asymptotes, equations (13a) and (13b). The maximum deviation is at the cross-over point ($Re_L \simeq 1000$) where their result is about 5 per cent above that due to Kutateladze [6]. The analysis of Kunz and Yerazunis [15] clearly encompasses that presented here but their improved equations are entirely too complicated for explicit algebraic expression.

It is emphasized that, beyond equation (5), no new information has been derived in this note. It is merely claimed that a convenient and generalized formulation has been presented for use in film flow problems in which a slight sacrifice in accuracy is acceptable in order to incorporate analytic solutions into subsequent steps.

CONCLUSIONS

It is concluded that all co-current thin film data can be reduced to a single function of one variable taking into account both interfacial shear and gravitational terms. This universal plot is susceptible to a very simple analysis, Moreover, the results are explicit in form for ease of incorporation into the transport equations for heat or mass transfer problems.

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